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A topological approach to canonical extensions in finitely generated varieties of lattice-based algebras

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ABSTRACT

This paper investigates completions in the context of finitely generated lattice-based varieties of algebras. It is shown that, for such a variety $\mathcal{A}$, the order-theoretic conditions of density and compactness which characterise the canonical extension of (the lattice reduct of) any $A \in \mathcal{A}$ have truly topological interpretations. In addition, a particular realisation is presented of the canonical extension of A; this has the structure of a topological algebra $n_{\mathcal{A}}(A)$ whose underlying algebra belongs to $\mathcal{A}$. Furthermore, each of the operations of $n_{\mathcal{A}}(A)$ coincides with both the σ-extension and the π-extension of the corresponding operation on A, with which a canonical extension is customarily equipped. Thus, in particular, the variety $\mathcal{A}$ is canonical, and all its operations are smooth. The methods employed rely solely on elementary order-theoretic and topological arguments, and bypass the subtle theory of canonical extensions that has been developed for lattice-based algebras in general.

1. Introduction

A *lattice-based algebra* is an algebraic structure which is a lattice equipped with a (possibly empty) set of additional operations. This paper investigates completions of algebras in finitely generated varieties of lattice-based algebras. We fix until further notice such a variety $\mathcal{A}$. Because $\mathcal{A}$ is finitely generated, there is a finite set $\mathcal{M}$ of finite algebras in $\mathcal{A}$ such that $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, that is, each $A \in \mathcal{A}$ embeds as a subalgebra of a product of algebras drawn from $\mathcal{M}$. This fact comes from two well-known theorems from universal algebra, namely Birkhoff's Subdirect Product Theorem and Jónsson's Lemma (see for example [2]). The representation of $\mathcal{A}$ as $\mathbb{ISP}(\mathcal{M})$ is fundamental to our approach: by equipping each $M \in \mathcal{M}$ with the discrete topology we arrive at a category $\mathcal{A}_{\mathcal{T}}$ of Boolean topological algebras within which our completions will live. (Here 'Boolean' refers to the topology rather than to the algebra: the underlying space of a Boolean topological algebra is compact and totally disconnected, and the operations of the algebra are continuous.) Because these completions are concretely built using $\mathcal{M}$, the assumption of finite generation is hardwired in from the start.

Our primary aim in writing this paper is to demonstrate to the canonical extensions fraternity how topological techniques can profitably be applied to the particular case of finitely generated lattice-based varieties. These techniques are different from, and more direct than, those specialising to this case the traditional methodology of canonical extensions as recently employed by Gehrke and Vosmaer [12, Section 4], [17] to study finitely generated varieties. We provide here very little background on canonical extensions. Nevertheless, it would be disingenuous to proceed straight to our results on completions without some brief remarks on how these results came about. Further detail on the context can be found in [12,5].

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Canonical extensions of lattice-based algebras are special lattice completions, equipped with suitable extensions of the non-lattice operations. They originated with Jónsson and Tarski [16] on Boolean algebras with operators (BAOs) (see [15] for a useful survey). A key aim of their pioneering work was to devise an algebraic means of analysing additional operations on Boolean algebras, by lifting these operations to the canonical extensions. This aim has remained a central plank of canonical extension theory as the scope of the theory has widened, to embrace distributive lattices with additional operations (Gehrke and Jónsson [9,10]) and lattice-based algebras (Gehrke and Harding [8]). The methodology is of most value for varieties which are *canonical*, that is, closed under the passage to canonical extensions. Canonicity is especially valuable when canonical extensions are used in the semantic modelling (both algebraic and relational) of logics (see [12] for a recent introductory account, and also, for example, [11,5] and the references therein).

In their paper [8], Gehrke and Harding employed the theory of Galois connections to define, and to demonstrate the existence (uniquely up to isomorphism) of, the canonical extension of a bounded lattice. The extension was thereby characterised by two order-theoretic properties, known as *density* and *compactness*. These properties (recalled in Section 2) specify how the lattice sits in its completion. There has been over the years an 'on-off' relationship between canonical extensions on the one hand and duality theory and topology on the other. The topological terminology, originating with Jónsson and Tarski, arose from the way that, in the Boolean case, the canonical extension was obtained with the aid of Stone's topological duality. Likewise, for algebras having a distributive lattice reduct, canonical extensions can be built via Priestley duality. Subsequently, Davey, Haviar and Priestley [4] gave an overtly topological interpretation of the density and compactness conditions in the distributive lattice case. In this paper we highlight the way in which these order-theoretic properties characterising canonical extensions are genuinely topological conditions also in the setting of finitely generated varieties of lattice-based algebras.

Drawing on ideas from natural duality theory, Davey, Gouveia, Haviar and Priestley [3] were able to show the existence of a *natural extension* $n_{\mathcal{B}}(B)$ for each algebra B in a prevariety $\mathcal{B} := \mathbb{ISP}(\mathcal{N})$, where $\mathcal{N}$ is *any* set of finite algebras of common type, not necessarily lattice-based. By giving each member of $\mathcal{N}$ the discrete topology, they obtained a category $\mathcal{B}_{\mathcal{T}}$ of Boolean topological algebras. By virtue of its construction, $n_{\mathcal{B}}(B)$ belongs to $\mathcal{B}_{\mathcal{T}}$ and it can easily be shown that the construction sets up a functor from $\mathcal{B}$ to $\mathcal{B}_{\mathcal{T}}$ [3, Proposition 3.2]. We can specialise to our finitely generated lattice-based variety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$. We show that the natural extension $n_{\mathcal{A}}(A)$ of $A \in \mathcal{A}$, which contains (an isomorphic copy of) A, supplies a realisation of the canonical extension of A; it is constructed as a suitable topologically closed sublattice of a product of copies of M_i for $M_i \in \mathcal{M}$ (see Theorems 2.4 and 3.3 below). Furthermore, $n_{\mathcal{A}}(A)$, as an algebra, necessarily belongs to $\mathcal{A}$.

For a lattice-based algebra in general, one customarily obtains a canonical extension by first forming the canonical extension of the underlying lattice and thereafter superimposing extensions of the non-lattice operations. Thanks to the density and compactness conditions, there are two natural ways to extend any map f from a lattice to its canonical extension, in the manner of an envelope built as a $\liminf$ or a $\limsup$; these extensions are denoted f^σ and f^π. But the natural extension of $A \in \mathcal{A}$ is already equipped with an extension of each algebraic operation f, since A is (isomorphic to) a subalgebra of $n_{\mathcal{A}}(A)$ in $\mathcal{A}$; moreover, since $n_{\mathcal{A}}(A)$ is a topological algebra, f is continuous on $n_{\mathcal{A}}(A)$. We show that f^σ and f^π both coincide with f on $n_{\mathcal{A}}(A)$ (see Theorem 3.5). Our proof of this fact uses an elementary topological argument involving convergence of filterbases (see for example [1] or [6] for the basic notions). This approach, which puts $\liminf$ and $\limsup$ constructions centre stage throughout, was suggested by the anonymous referee of a different, but related, paper. We gratefully acknowledge that referee's contribution to the present paper.

We note that, while our account gives a geodesic route to the results advertised above, it by no means covers all that can usefully be said about canonical extensions of algebras in finitely generated varieties of lattice-based algebras. Specifically, we do not discuss here the structure of these extensions viewed as freestanding topological algebras. This topic is pursued in our companion paper [5, Section 3].

2. Complete sublattices of products of finite lattices

In this section we investigate completions of lattices which can be identified with sublattices of arbitrary products of finite lattices. We defer until later consideration of other algebraic operations which may be present.

First of all we briefly recall some basic definitions from the theory of canonical extensions. Let L be a sublattice of a complete lattice C. Then C is called a *completion* of L. (More generally, if $e\colon L \to C$ is an embedding of the lattice L into the complete lattice C, then the pair (e, C) is also called a *completion* of L.) Write $T \Subset S$ to mean that T is a finite subset of S. A completion C of L is said to be *dense* if every element of C can be expressed both as a join of meets and as a meet of joins of elements of L. In addition, C is called a *compact* completion of L if, for any down-directed subset A and every up-directed subset B of L with $\bigwedge A \leqslant \bigvee B$, we have $a \leqslant b$, for some $a \in A$ and $b \in B$. (Note that, by definition, directed subsets are non-empty.) Equivalent formulations of the compactness condition are available, but the one we give is convenient for our purposes. A *canonical extension* of a lattice L is a completion C of L that is both dense and compact. Gehrke and Harding [8] proved that every bounded lattice L has a canonical extension and that any two canonical extensions of L are isomorphic via an isomorphism that fixes the elements of L.

Now let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ be the quasivariety generated by $\mathcal{M}$, where $\mathcal{M}$ is a finite set of finite lattice-based algebras. (So the algebras in $\mathcal{M}$, and therefore those in $\mathcal{A}$, are of the form $\langle A; \vee, \wedge, F\rangle$, for some set F of operations, with the reduct $\langle A; \vee, \wedge\rangle$ a lattice.) Our aim will be to recognise suitable subalgebras of products of algebras in $\mathcal{M}$ as candidates for the canonical extensions of algebras on $\mathcal{A}$. So we shall begin with some generalities concerning complete sublattices of products of finite lattices and the way in which topology and lattice structure interact on such objects. (We recall that a non-empty subset L of a complete lattice K is called a *complete sublattice* of K if it is closed under joins and meets (taken in K) of arbitrary non-empty subsets.)

Our first result, Proposition 2.1 below, generalises [4, Lemma 2.2]. It shows that, under rather general conditions, the lattice-theoretic density condition equates to a condition of topological density. In preparation, we note the following well-known description of the closure in topological products. Let $\{M_s\}_{s\in S}$ be a family of topological spaces indexed by a non-empty set S. Let L be a subset of $\prod_{s\in S} M_s$. An element x of $\prod_{s\in S} M_s$ is *locally in* L if, for every $T \Subset S$, there exists $a \in L$ with $x{\restriction}_T = a{\restriction}_T$. We denote by $\mathrm{loc}(L)$ the set of all elements of $\prod_{s\in S} M_s$ that are locally in L. When each M_s is finite and endowed with the discrete topology, $\mathrm{loc}(L)$ is the topological closure of L in $\prod_{s\in S} M_s$. For each $x \in \mathrm{loc}(L)$ and each T with $T \Subset S$, we define

$$\mathcal{B}_{x,T} := \{a \in L \mid x{\restriction}_T = a{\restriction}_T\}$$

and let $\mathcal{B}_x := \{\mathcal{B}_{x,T} \mid T \Subset S\}$. Since each set $\mathcal{B}_{x,T}$ is non-empty and the family $\mathcal{B}_x$ is closed under finite intersections, this family is a filterbase on L.

Now assume that each M_s is a complete lattice, so that $\prod_{s\in S} M_s$ is a complete lattice when joins and meets are calculated pointwise. Let L be a sublattice of $\prod_{s\in S} M_s$. Then for $x \in \mathrm{loc}(L)$ and $T \Subset S$ we define

$$x_T := \bigwedge \mathcal{B}_{x,T} \quad \text{and} \quad x^T := \bigvee \mathcal{B}_{x,T}.$$

Proposition 2.1. *Let S be a non-empty set, let M_s be a complete lattice, for all $s \in S$, and let L be a sublattice of $M := \prod_{s\in S} M_s$.*

(i) *Let $x \in \mathrm{loc}(L)$.*
  (a) $x = \bigvee_{T\Subset S} \bigwedge \mathcal{B}_{x,T} = \bigwedge_{T\Subset S} \bigvee \mathcal{B}_{x,T}$ *with the join and the meet, respectively, up-directed and down-directed*;
  (b) $\mathcal{B}_{x,T} = [x_T, x^T] \cap L$, *for each T with $T \Subset S$.*

(ii) *Assume in addition that each M_s is finite. Then*
  (a) $\mathrm{loc}(L)$ *is a complete sublattice of M and is the complete sublattice generated by L*;
  (b) $\mathrm{loc}(L)$ *is a dense completion of L.*

(iii) *Assume that each M_s is a finite lattice equipped with the discrete topology and $M = \prod_{s\in S} M_s$ with the product topology. Then, for each $x \in \mathrm{loc}(L)$, the filterbase $\mathcal{B}_x$ converges to x.*

Proof. Consider (i)(a). Fix an element x that is locally in L. Then $x_T{\restriction}_T = x{\restriction}_T$. Clearly, $\bigvee\{x_T \mid T \Subset S\} \geqslant x$, so to prove equality it remains to show that $x_T \leqslant x$, for all $T \Subset S$. Let $T \Subset S$ and $s \in S$. Then $x{\restriction}_{T\cup\{s\}} = a{\restriction}_{T\cup\{s\}}$, for some $a \in L$. It follows that $x_T \leqslant a$, whence $x_T(s) \leqslant a(s) = x(s)$. Thus, $x_T \leqslant x$, as required. Hence $x = \bigvee\{x_T \mid T \Subset S\}$ and the join is clearly directed. The second assertion in (i)(a) follows by order duality. We have already observed that x_T and x^T belong to $\mathcal{B}_{x,T}$. Since, for all $t \in T$, we have $x_T(t) = x(t) = x^T(t)$, it follows that $[x_T, x^T] \cap L \subseteq \mathcal{B}_{x,T}$. The reverse inclusion follows from the definition of x_T and x^T.

Now assume that each M_s is a finite lattice. That $\mathrm{loc}(L)$ forms a complete sublattice of $\prod_{s\in S} M_s$ will follow easily once we prove that (with pointwise joins and meets):

$$\varnothing \neq A \subseteq L \implies \bigvee A \in \mathrm{loc}(L) \quad \text{and} \quad \bigwedge A \in \mathrm{loc}(L). \tag{$*$}$$

Let A be a non-empty subset of L and let $x := \bigvee A$. Let $T \Subset S$ and let $t \in T$. Then, since M_t is finite,

$$x(t) = \bigvee_{a\in A} a(t) = a_1^t(t) \vee \cdots \vee a_{j_t}^t(t),$$

for some $j_t \in \mathbb{N}$ and $a_1^t, \ldots, a_{j_t}^t \in A$. Define

$$a := \bigvee\bigl\{a_1^t \vee \cdots \vee a_{j_t}^t \bigm| t \in T\bigr\}.$$

Then $a \in L$ and $a \leqslant \bigvee A = x$. We have $a(t) \geqslant a_1^t(t) \vee \cdots \vee a_{j_t}^t(t) = x(t)$, for each $t \in T$. Thus, $x{\restriction}_T = a{\restriction}_T$. So $\bigvee A \in \mathrm{loc}(L)$, and $\bigwedge A \in \mathrm{loc}(L)$ by duality. By replacing L by $\mathrm{loc}(L)$ in $(*)$, we conclude at once that $\mathrm{loc}(L)$ is a complete sublattice of $\prod_{s\in S} M_s$. This proves (ii)(a). Part (ii)(b) follows immediately from (i)(a) and the definition of density.

Consider (iii). We require to show that any basic open neighbourhood U of $x \in \mathrm{loc}(L)$ contains a member of $\mathcal{B}_x$. Since the topology on each M_s is discrete we may assume that U is of the form

$$U = \prod_{s\in S} U_s \quad \text{where } U_s = \begin{cases} M_s & \text{if } s \notin T, \\ \{x(s)\} & \text{if } s \in T, \end{cases}$$

for some $T \Subset S$. It is immediate that $\mathcal{B}_{x,T} \subseteq U$. $\square$

Proposition 2.1 relates lattice-theoretic properties of $C := \mathrm{loc}(L)$ to topological properties. In particular $\mathrm{loc}(L)$ serves as a dense completion of L, with density witnessed, both topologically and order-theoretically, in a very special way. For $x \in C$ we have

$$x = \bigvee_{T \Subset S} \bigwedge\big(\big[x_T, x^T\big] \cap L\big) = \bigwedge_{T \Subset S} \bigvee\big(\big[x_T, x^T\big] \cap L\big).$$

Restated in the notation of lim inf and lim sup, as defined in a complete lattice, this becomes $x = \liminf \mathcal{B}_x = \limsup \mathcal{B}_x$, where $\mathcal{B}_x$ is the filterbase of subsets $\mathcal{B}_{x,T} := [x_T, x^T] \cap L$, for $T \Subset S$. Furthermore, we have seen that $\mathcal{B}_x$ converges to x (since C is closed in $\prod_{s\in S} M_s$, we do not need to distinguish between convergence in C and in the full product). We shall exploit these observations in Section 3.

Our next task is to investigate the compactness property demanded of a canonical extension, again working in products of finite lattices.

Lemma 2.2. *Let S be a non-empty set, let M_s be a finite lattice, for all $s \in S$, and let L be a sublattice of $\prod_{s\in S} M_s$. Let A be a down-directed set and B an up-directed set in L. Then*

$$\bigwedge A \nleqslant \bigvee B \quad \iff \quad (\exists z \in S)\ (\forall a \in A)\ (\forall b \in B) \quad a(z) \nleqslant b(z).$$

Proof. Let π_s be the natural projection from $\prod_{s\in S} M_s$ onto M_s. Note that $\pi_s(A)$ and $\pi_s(B)$ are, respectively, down-directed and up-directed in M_s. Since M_s is finite, $\pi_s(A)$ has a least element and $\pi_s(B)$ has a greatest element. This fact is used to justify the last equivalence below.

$$\begin{aligned}
\bigwedge A \nleqslant \bigvee B \quad &\iff \quad (\exists z \in S) \quad \Big(\bigwedge A\Big)(z) \nleqslant \Big(\bigvee B\Big)(z) \\
&\iff \quad (\exists z \in S) \quad \bigwedge_{a\in A} a(z) \nleqslant \bigvee_{b\in B} b(z) \\
&\iff \quad (\exists z \in S)\ (\forall a \in A)\ (\forall b \in B) \quad a(z) \nleqslant b(z). \qquad \square
\end{aligned}$$

So far, we have worked with an arbitrary product of finite lattices. We shall need to specialise to the situation where our product lattices are of the form $M_1^{S_1} \times \cdots \times M_\ell^{S_\ell}$, where $\mathcal{M} = \{M_1, \ldots, M_\ell\}$ is a finite family of finite lattices. The next result, a technical lemma, will allow us to reduce to the case $\ell = 1$.

Lemma 2.3. *Let $X_1, \ldots, X_\ell$ be complete lattices, let L be a sublattice of the product $X_1 \times \cdots \times X_\ell$ and assume that X_i is a compact completion of $\pi_i(L)$, for each $i \in \{1, \ldots, \ell\}$. Then $X_1 \times \cdots \times X_\ell$ is a compact completion of L.*

Proof. Let $A, B \subseteq L$, with A down-directed and B up-directed. Then

$$\begin{aligned}
\bigwedge A \leqslant \bigvee B \quad &\Longrightarrow \quad (\forall i) \quad \bigwedge \pi_i(A) \leqslant \bigvee \pi_i(B) \quad \text{in } \pi_i(L) \\
&\Longrightarrow \quad (\forall i)\ \big(\exists a^i \in A\big)\ \big(\exists b^i \in B\big) \quad \pi_i\big(a^i\big) \leqslant \pi_i\big(b^i\big),
\end{aligned}$$

as X_i is a compact completion of $\pi_i(L)$. Now we can find $a \in A$ and $b \in B$ with $a \leqslant a^1 \wedge \cdots \wedge a^\ell$ and $b \geqslant b^1 \vee \cdots \vee b^\ell$. Then

$$a \leqslant \big(\pi_1\big(a^1\big), \ldots, \pi_\ell\big(a^\ell\big)\big) \leqslant \big(\pi_1\big(b^1\big), \ldots, \pi_\ell\big(b^\ell\big)\big) \leqslant b. \qquad \square$$

Given topological spaces Z and M, we denote by $\mathcal{C}(Z, M)$ the set the continuous functions from Z to M. If M is a topological algebra, then $\mathcal{C}(Z, M)$, when endowed with the pointwise operations, becomes an algebra of the same type as M.

Theorem 2.4. *Let $Z_1, \ldots, Z_\ell$ be compact spaces, let $M_1, \ldots, M_\ell$ be finite lattices each equipped with the discrete topology and let L be a sublattice of the product $\mathcal{C}(Z_1, M_1) \times \cdots \times \mathcal{C}(Z_\ell, M_\ell)$.*

(i) *The lattice $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ is a compact completion of L.*
(ii) *The topological closure of L in the product $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ is a canonical extension of L.*

Proof. For (i), it suffices, by Lemma 2.3, to consider the case that $\ell = 1$. Let L be a sublattice of the lattice $\mathcal{C}(Z, M)$ of continuous functions from Z into M, for some compact topological space Z and some finite lattice M. Let A, B be subsets of L with A down-directed and B up-directed, and $\bigwedge A \nleqslant \bigvee B$. We must prove that $a \nleqslant b$ for every $a \in A$ and $b \in B$. As M is finite, by Lemma 2.2 it suffices to show that there exists $z \in Z$ such that $a(z) \nleqslant b(z)$, for all $a \in A$ and all $b \in B$. For $a, b \in \mathcal{C}(Z, M)$, define

$$[\![a \nleqslant b]\!] := \{z \in Z \mid a(z) \nleqslant b(z)\}.$$

Note that $[\![a \nleqslant b]\!]$ is the complement in Z of the equaliser of the continuous maps $a \wedge b$ and a. Since M is finite and has the discrete topology, it follows that $[\![a \nleqslant b]\!]$ is a clopen subset of Z, and $\bigwedge A \nleqslant \bigvee B$ guarantees that $[\![a \nleqslant b]\!]$ is non-empty, for all $a \in A$ and all $b \in B$. Define

$$\mathcal{F} := \{[\![a \nleqslant b]\!] \mid a \in A \text{ and } b \in B\}.$$

For any $a_1, a_2 \in A$ and any $b_1, b_2 \in B$, we can choose $a_0 \in A$ and $b_0 \in b$ such that $a_0 \leqslant a_1 \wedge a_2$ and $b_0 \geqslant b_1 \vee b_1$. It is easily seen that

$$[\![a_0 \nleqslant b_0]\!] \subseteq [\![a_1 \nleqslant b_1]\!] \cap [\![a_2 \nleqslant b_2]\!].$$

Hence, $\mathcal{F}$ is a filterbase of closed subsets of Z. The compactness of Z guarantees that $\bigcap \mathcal{F}$ is non-empty. Choose $z \in \bigcap \mathcal{F}$, then $a(z) \nleqslant b(z)$, for all $a \in A$ and all $b \in B$, as required. This completes the proof of (i).

For (ii) we first note that the topological closure $\mathrm{loc}(L)$ of L in the product $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ is a complete sublattice of this product, by Proposition 2.1(ii). It follows immediately from (i) that $\mathrm{loc}(L)$ is a compact completion of L. By Proposition 2.1(i), $\mathrm{loc}(L)$ is also a dense completion of L. □

3. Canonical extensions in the context of finitely generated lattice-based varieties

We initially consider a class $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a non-empty set of finite lattice-based algebras of common type; the assumption that $\mathcal{M}$ is finite will be added later. Let $A \in \mathcal{A}$. Then we can regard A as a subalgebra of a product $\prod_{s \in S} M_s$, where each $M_s \in \mathcal{M}$. Equip each M_s with the discrete topology and let $C = \mathrm{loc}(A)$, the topological closure of A in $\prod_{s \in S} M_s$. The operations are given pointwise on $\prod_{s \in S} M_s$, and on its subalgebras. The following result is elementary.

Proposition 3.1. *Let $\mathcal{M}$ be a set of finite, discretely topologised, lattice-based algebras (of the same type). Assume that C is the topological closure of a subalgebra of a product of algebras from $\mathcal{M}$. Then C is a compact topological algebra with respect to the induced product topology.*

We now investigate more closely how the non-lattice operations of A relate to those of C. Consider any unary operation f in the type of the algebras in $\mathcal{M}$. (Operations of other arities can be handled similarly, at the expense only of more complicated notation.) Then, by Proposition 3.1, (the interpretation of) f on C is continuous. We shall invoke the results of the previous section, applied with L as (the lattice reduct of) A. With $\mathcal{B}_x$ as defined there, $f(\mathcal{B}_x)$ converges to $f(x)$. Observe that since the operations $\bigvee$ and $\bigwedge$ are defined pointwise, so too are the lim inf and lim sup in terms of which $x \in C$ can be expressed.

Lemma 3.2. *Let $\mathcal{M}$ be a set of finite, discretely topologised, lattice-based algebras (of the same type) and let f be any unary operation in the type. Assume that C is the topological closure of a subalgebra A of a product $\prod_{s \in S} M_s$ of algebras from $\mathcal{M}$. Let $x \in C$ and let $\mathcal{B}_x$ be the filterbase defined above. Then*

$$f(x) = \liminf f(\mathcal{B}_x) = \limsup f(\mathcal{B}_x).$$

Proof. We already know that $x = \liminf \mathcal{B}_x = \limsup \mathcal{B}_x$ and that $\mathcal{B}_x$ converges to x. Consider a fixed $s \in S$. Since the filterbase $\pi_s(\mathcal{B}_x)$ converges to $\pi_s(x)$ and the topology on M_s is discrete, there exists some $T_s \Subset S$ such that $\pi_s([x_T, x^T] \cap A) = \{\pi_s(x)\}$ for $T_s \subseteq T \Subset S$. Therefore, since $f(\mathcal{B}_x)$ converges to $f(x)$, the singleton set $\pi_s(f([x_T, x^T] \cap A))$ equals $\{\pi_s(f(x))\}$. It follows that, for each s, we have $\pi_s(f(x)) = \liminf \pi_s(f(\mathcal{B}_x)) = \limsup \pi_s(f(\mathcal{B}_x))$. Hence $f(x) = \liminf f(\mathcal{B}_x) = \limsup f(\mathcal{B}_x)$. □

Thus far in this section we have considered only dense completions. Our primary interest, however, is in canonical extensions. Theorem 2.4 shows how such completions can arise. In order to apply it to algebras A in $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where now $\mathcal{M} = \{M_1, \ldots, M_\ell\}$ (a finite set), we need to show that each A can be represented as a subalgebra of a product of powers of the algebras M_i, with suitable spaces of continuous maps on compact spaces as the exponents. Strongly motivated by the theory of natural dualities, we shall achieve this, as in [3], by taking $Z_i := \mathcal{A}(A, M_i)$, for $i = 1, \ldots, \ell$. Here the underlying set of Z_i is the set of homomorphisms from A into M_i and Z_i is endowed with the subspace topology derived from the power M_i^A, where M_i carries the discrete topology. Since Z_i is a closed subspace of the product, it is compact.

We embed A into $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ by means of the map

$$e_A : A \to \prod_{1 \leqslant i \leqslant \ell} M_i^{\mathcal{A}(A, M_i)}$$

given by $e_A(a)(i)(x) = x(a)$, for $i \in \{1, \ldots, \ell\}$ and $x \in \mathcal{A}(A, M_i)$; we call the map $e_A(a)$, for $a \in A$, a *multisorted evaluation map*. The map e_A is a homomorphism and, because $A \in \mathbb{ISP}(\mathcal{M})$, it is also an embedding. Since each map $e_A(a)$ is continuous we can restrict the codomain of e_A and write

$$e_A : A \to \prod_{1 \leqslant i \leqslant \ell} \mathcal{C}(Z_i, M_i).$$

Following the notation and terminology of Davey et al. [3], we define the *natural extension* $n_{\mathcal{A}}(A)$ of A (relative to $\mathcal{M} = \{M_1, \ldots, M_\ell\}$) to be the topological closure of $e_A(A)$ in $\prod_{1 \leqslant i \leqslant \ell} M_i{}^{\mathcal{C}(Z_i, M_i)}$. We observe that $e_A(A)$ is by construction a subalgebra of $\prod_{1 \leqslant i \leqslant \ell} \mathcal{C}(Z_i, M_i)$. We identify A with $e_A(A)$ and so regard A as a subalgebra of $n_{\mathcal{A}}(A)$, with the operations in each case being determined pointwise. (It is known that the algebra $n_{\mathcal{A}}(A)$ is independent of the choice of generating set $\mathcal{M}$, for the variety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ [3, Corollary 3.7].)

The following theorem is a corollary of Theorem 2.4(ii).

Theorem 3.3. *Let $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ where $\mathcal{M}$ is a finite set of finite lattice-based algebras (of the same type). Then, for each $A \in \mathcal{A}$, the lattice reduct of the natural extension $n_{\mathcal{A}}(A)$ is a dense and compact completion of the lattice reduct of A.*

The canonical extension of a lattice is unique, up to an isomorphism fixing that lattice. This allows canonical extensions to be analysed abstractly without reference to any particular construction. Theorem 3.3 implies that, for any finitely generated variety $\mathcal{A}$ of lattice-based algebras, we have a dense completion for each $A \in \mathcal{A}$ which is a closed subalgebra of a product of powers of discretely topologised finite algebras and which is, in addition, a compact completion. Working with this particular concrete representation of the canonical extension, rather than the abstract characterisation, has significant merits, because the completion operates, ab initio, as a completion for the algebras rather than merely their lattice reducts, and incorporates topological structure as well.

Let us view the natural extension $n_{\mathcal{A}}(A)$ of an algebra $A \in \mathcal{A}$ as providing, at the lattice level, a canonical extension of $\mathcal{A}$, then an obvious question arises. How, for each non-lattice operation f in the type, does f on $n_{\mathcal{A}}(A)$ relate to the σ- and π-extensions of f from A to $n_{\mathcal{A}}(A)$? For a general lattice-based variety (that is, one that is not necessarily finitely generated), the density and compactness properties are used in conjunction to derive even the most basic properties of maps f^σ and f^π on the canonical extension. It is therefore not surprising that our reconciliation of f with f^σ and f^π brings the compactness of the completion into play. However we emphasise that our proof relies very directly on the definitions and does not need to call on the theory of canonical extensions in general.

We fix $A \in \mathcal{A}$ and denote $n_{\mathcal{A}}(A)$ by C. As before, we may, and shall, restrict attention to algebraic operations which are unary. Let f be such an operation. Spelling out the conclusion of Lemma 3.2 in alternative notation, we have, for all $x \in n_{\mathcal{A}}(A)$,

$$f(x) = \begin{cases} \bigvee_{T \Subset S} \bigwedge \{f(a) \mid a \in A \text{ and } x_T \leqslant a \leqslant x^T\}, \\ \bigwedge_{T \Subset S} \bigvee \{f(a) \mid a \in A \text{ and } x_T \leqslant a \leqslant x^T\}. \end{cases}$$

These formulae may be compared with those defining $f^\sigma(x)$ and $f^\pi(x)$ for $x \in C$:

$$f^\sigma(x) := \bigvee \Big\{ \bigwedge \big\{ f(a) \bigm| a \in A \text{ and } p \leqslant a \leqslant q \big\} \bigm| p \in K(C),\ q \in O(C) \text{ and } p \leqslant x \leqslant q \Big\},$$

$$f^\pi(x) := \bigwedge \Big\{ \bigvee \big\{ f(a) \bigm| a \in A \text{ and } p \leqslant a \leqslant q \big\} \bigm| p \in K(C),\ q \in O(C) \text{ and } p \leqslant x \leqslant q \Big\},$$

where $K(C)$ and $O(C)$ are, respectively, the filter elements and the ideal elements of C, so that $p \in K(C)$ if and only if $p = \bigwedge P$, for some P with $\varnothing \neq P \subseteq A$ and $q \in O(C)$ if and only if $q = \bigvee Q$, for some Q with $\varnothing \neq Q \subseteq A$. Here we may without loss of generality restrict P to be down-directed and Q to be up-directed, since A is closed under finite meets and joins. It is immediate that each of f^σ and f^π extends f, since each element of A is in $K(C) \cap O(C)$ (see Gehrke and Harding [8, Lemma 4.2(i)]).

In [8, Lemma 3.3], Gehrke and Harding show that, in the canonical extension C of any bounded lattice, the ideal and filter elements form sublattices of C. Their proof relies on a restricted form of distributivity valid in canonical extensions in general [8, Lemma 3.2]. They then show easily, exploiting compactness of the completion, that $f^\sigma \leqslant f^\pi$ for any $f : A \to A$ [8, Lemma 4.2(ii)]. In keeping with our thesis that the finitely generated case can be treated without recourse to the full-blown theory of canonical extensions, we include a direct proof of the following lemma.

Lemma 3.4. *Let $A \in \mathcal{A}$ and $C = n_{\mathcal{A}}(A)$ and let f be as above. Then, for each $x \in C$,*

$$\mathcal{B}'_x := \big\{ [p, q] \cap A \bigm| p \in K(C),\ q \in O(C) \text{ and } p \leqslant x \leqslant q \big\}$$

is a filterbase of subsets of A with $\mathcal{B}'_x \supseteq \mathcal{B}_x$.

Proof. First we confirm that the members of $\mathcal{B}'_x$ are non-empty. Let $p \leqslant x \leqslant q$, where $p \in K(C)$ and $q \in O(C)$. Let $p = \bigwedge P$ and $q = \bigvee Q$, where the meet and join are, respectively, down-directed and up-directed. By compactness of the completion, there exist $a \in P$ and $b \in Q$ such that $a \leqslant b$. Necessarily $p \leqslant a \leqslant b \leqslant q$ so $[p,q] \cap A \neq \varnothing$.

Now let $p_1, p_2 \in K(C)$ and $q_1, q_2 \in O(C)$, with $p_i \leqslant x \leqslant q_i$ $(i = 1,2)$. We shall show that there exist $p \in K(C)$ and $q \in O(C)$ with $p_1 \vee p_2 \leqslant p \leqslant q \leqslant q_1 \wedge q_2$. We may assume that $q_i = \bigvee Q_i$, where Q_i is up-directed $(i = 1,2)$. For any $i = 1,2$ and any $s \in M_s$, the set $\pi_s(Q_i)$ is up-directed in M_s, and hence has a greatest element, $m_{i,s}$, say. Choose $a_{i,s} \in Q_i$ such that $\pi_s(a_{i,s}) = m_{i,s}$. Let $a_s = a_{1,s} \wedge a_{2,s}$, for $s \in S$ and let $q = \bigvee_{s \in S} a_s$, so that $q \in O(C)$. We have, because joins and meets are calculated pointwise,

$$\pi_s(x) \leqslant \pi_s\Bigl(\bigvee Q_1\Bigr) \wedge \pi_s\Bigl(\bigvee Q_2\Bigr) = \pi_s(a_{1,s}) \wedge \pi_s(a_{2,s}) = \pi_s(a_{1,s} \wedge a_{2,s}) = \pi_s(a_s) \leqslant \pi_s(q),$$

so that $x \leqslant q$. Also, since $a_s \leqslant a_{i,s}$, we have

$$\pi_s(a_s) \leqslant \pi_s(a_{i,s}) = m_{i,s} = \bigvee \pi_s(Q_i) = \pi_s\Bigl(\bigvee Q_i\Bigr).$$

Therefore $a_s \leqslant \bigvee Q_i = q_i$, for all $s \in S$ and $i = 1,2$. We conclude that $q \leqslant q_1$ and $q \leqslant q_2$. We can construct p likewise. We now have that $x \in [p,q]$ and

$$[p,q] \cap A \subseteq \bigl([p_1,q_1] \cap A\bigr) \cap \bigl([p_2,q_2] \cap A\bigr).$$

We have shown that $\mathcal{B}'_x$ is indeed a filterbase at x. It contains $\mathcal{B}_x$ because $x_T \in K(C)$ and $x^T \in O(C)$ whenever $T \Subset S$. $\square$

Our final theorem, which incorporates and summarises our results so far, is now virtually immediate. In the statement we do not distinguish notationally between an operation in the type of $\mathcal{A}$ and its interpretation on a given member of $\mathcal{A}$.

Theorem 3.5. *Let $\mathcal{A}$ be a finitely generated variety of lattice-based algebras, represented as $\mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras. Let $A \in \mathcal{A}$ and let $n_{\mathcal{A}}(A)$ be the natural extension of A.*

(i) *$n_{\mathcal{A}}(A)$ is a Boolean topological algebra whose underlying lattice is a canonical extension of A. In addition the underlying algebra of $n_{\mathcal{A}}(A)$ belongs to $\mathcal{A}$.*
(ii) *For each operation f in the type of the algebras each of the extensions f^σ and f^π of f interpreted on A is equal to f interpreted on $n_{\mathcal{A}}(A)$.*

In particular, $\mathcal{A}$ is canonical, and each operation f in the type of $\mathcal{A}$ is smooth, in the sense that $f^\sigma = f^\pi$.

Proof. Only the first assertion in (ii) remains to be proved. Let $x \in C := n_{\mathcal{A}}(A)$. Since $\mathcal{B}'_x$, as defined in Lemma 3.4, refines $\mathcal{B}_x$, and $f(\mathcal{B}_x)$ converges to $f(x)$, it is immediate that $f(\mathcal{B}'_x)$ converges to $f(x)$ too. We can now argue exactly as we did to prove that $f(x) = \liminf f(\mathcal{B}_x) = \limsup f(\mathcal{B}_x)$ to obtain likewise that $f(x) = \liminf f(\mathcal{B}'_x) = \limsup f(\mathcal{B}'_x)$. Since, by definition, $f^\sigma(x) = \liminf f(\mathcal{B}'_x)$ and $f^\pi(x) = \limsup f(\mathcal{B}'_x)$, we have $f(x) = f^\sigma(x) = f^\pi(x)$, as required.

Alternatively we may proceed as follows. It is a consequence of compactness and the fact that $\mathcal{B}'_x$ is a filterbase that $\liminf f(\mathcal{B}'_x) \leqslant \limsup f(\mathcal{B}'_x)$ (cf. the proof of Lemma 4.2 in [8]). Since $\mathcal{B}_x \subseteq \mathcal{B}'_x$, we have $\liminf f(\mathcal{B}'_x) \geqslant \liminf f(\mathcal{B}_x)$ and $\limsup f(\mathcal{B}'_x) \leqslant \limsup f(\mathcal{B}_x)$. As we already know that $\liminf f(\mathcal{B}_x) = f(x) = \limsup f(\mathcal{B}_x)$, we obtain $f^\sigma(x) = f(x) = f^\pi(x)$. $\square$

We conclude with some remarks concerning the relationship between topological convergence and order-convergence. Re recall that a filter $\mathcal{F}$ in a complete lattice X order-converges to a point x if

$$x = \sup_{F \in \mathcal{F}} \inf F = \inf_{F \in \mathcal{F}} \sup F.$$

We emphasise that the approach we have adopted in this paper depends critically on the fact that our algebras can be realised as subalgebras of products of *finite* algebras, with pointwise operations. As a consequence, liminfs and limsups of filterbases are formed in a way which, coordinatewise, is highly special: the associated nets are eventually constant. The topological structure in ordered topological spaces in general need not interact well with liminfs and limsups (indeed, order-convergence need not correspond to convergence with respect to any topology); see for example [13, II.1 and III.3] and also [7] and [14, Section 2]. In particular, order-convergence and topological convergence do not coincide on a complete lattice which fails to be meet- and join-continuous. There exist bounded lattices whose canonical extensions fail to be meet-continuous, and so are not continuous lattices (see [12, Example 3.1]). The point we are making is that Theorem 3.5, and the way we have arrived at it, are inextricably linked to our assumption that our variety of algebras is finitely generated. Nevertheless, we should emphasise that upper and lower envelopes, viewed topologically, do play a valuable role in the general theory of canonical extensions. This is demonstrated by the analysis of topological properties, with respect to a variety of different topologies, of the σ- and π-extensions of maps. This was initiated for the case of distributive lattices by Gehrke and Jónsson [10] and has recently been successfully extended to arbitrary bounded lattices by Vosmaer [17]. Of particular importance is the δ-topology (called the σ-topology in [10]); this is defined in terms of intervals $[p, q]$, for p a filter element and q an ideal element of the canonical extension; it is not defined solely in terms of the lattice order. The δ-topology in general does not coincide with intrinsic topologies available on arbitrary complete lattices, such as the interval topology or the bi-Scott topology. In bare outline, what happens in the finitely generated case is that various topologies coalesce and agree with the induced product topology with which we have worked throughout this paper. For varieties which are not finitely generated, different preservation properties of maps f^σ and f^π are captured in terms of conditions involving a plethora of non-coincident topologies (cf. [10, Theorem 2.27]).

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